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Degeneration through coalescence of the q-Painlevé VI equation

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Abstract. Starting from the q-discrete form of the Painlevé VI equation we obtain its degenerate forms by applying the procedure of coalescence of singularities. The whole cascade of degenerate forms is thus obtained leading to new forms for the discrete Painlevé IV and V equations. The Lax pairs of these discrete Painlevé equations are explicitly constructed, thus confirming their integrability.

The discovery of the q-discrete form of the Painlevé VI equation by one of us (HS) in collaboration with Jimbo [1] has filled a gap in the domain of discrete Painlevé equations ($\mathbb{P}$'s). With this discrete equation available, we were (at last) able to produce a discrete form for every single one of the $\mathbb{P}$'s. However, the importance of this mapping goes beyond the domain of pure taxonomy of the discrete $\mathbb{P}$'s. The fact that this equation is of q-type rather than of difference type (with the independent variable entering in a multiplicative rather than in an additive way) makes it a realization of a q-Garnier system [2] (in fact, the simplest, non-trivial one). Despite the fact that the discovery of q-P_{VI} is very recent, this equation has already been the object of several detailed studies. In [1] the linearization of this equation was presented through the explicit calculation of the corresponding Lax pair. In [3] the relation of this equation to the 'asymmetric' q-P_{III} has been established and the Schlesinger transforms of both q-P_{III} and q-P_{VI} were derived. The special solutions of q-P_{VI} in the form of a Casorati determinant of hypergeometric functions were given in [4]. Finally, in [5] the unified description of q-P_{VI} and its Schlesinger transformations (what two of the present authors [6] have dubbed the 'Grand Scheme') was presented. In this approach the bilinear form of q-P_{VI} is given in terms of a single τ-function that lives in a five-dimensional lattice. The dynamic equation has the form of a (non-autonomous) Hirota–Miwa equation and describes the evolution in the direction of the independent variables as well as in the direction of the parameters (under the action of the Schlesinger transforms).

In this paper we shall present yet another application of the q-P_{VI} equation. It is in fact known that the Painlevé equations can be used in order to produce more equations of the same kind. Some of the procedures are already known for the continuous $\mathbb{P}$'s [7, 8]: their Schlesinger transforms can define mappings that are nothing but discrete $\mathbb{P}$'s. It goes without saying that this approach can be (and has been [9]) extended to the discrete case in a straightforward way. Some procedures are characteristic of the discrete ones: the degeneracy concept we introduced in [10] is such an example. In this approach one considers the autonomous limit of the discrete $\mathbb{P}$, introduces some assumption of factorization and simplification, and then deautonomizes the obtained form. Finally, some procedures are from the outset common to both continuous and discrete $\mathbb{P}$'s. In this class we find the procedures based on limits or coalescences. The latter, which will be at the core of this paper, obtain degenerate (or, equivalently, non-generic) forms of $\mathbb{P}$'s through a coalescence of their singularities. (This, of course, explains the title of the paper. However, we must point out that at least two of us (BG, AR) use the not quite appropriate term 'coalescences' to denote the degenerate forms themselves. This terminology has the merit of avoiding confusion with the above-mentioned degeneracy concept).

Both continuous and discrete Painlevé equations are linked through a coalescence cascade in the following pattern:

$$P_{VI} \to P_V \to \{P_{IV}, P_{III}\} \to P_{II} \to P_I.$$

In the case of discrete equations the coalescence process is particularly interesting since there exist several forms for each discrete $\mathbb{P}$ (i.e. several discrete equations having the same $\mathbb{P}$ as continuous limit). Thus, this method can be used in order to generate new d-$\mathbb{P}$'s. In what follows we shall explore systematically the degeneration through coalescence of q-P_{VI}.

We start from the q-discrete form of the Painlevé VI equation:

$$y\underline{y} = \frac{(z-a)(z-b)}{(z-c)(z-1/c)} \tag{1a}$$

$$z\bar{z} = \frac{(y-p)(y-q)}{(y-r)(y-1/r)} \tag{1b}$$

where $y = y(n)$, $\bar{y} = y(n+1)$, $\underline{y} = y(n-1)$ (similarly for z) and $a = a_0\lambda^n$, $b = b_0\lambda^n$, $p = p_0\lambda^{n+1/2}$, $q = q_0\lambda^{n+1/2}$. The quantities c, r are constant and, moreover, the following constraint must hold: $a_0b_0 = p_0q_0$. The first coalescence is introduced through the change of variable $y = 1 + \delta\eta$ and the limit $\delta \to 0$. The parameters of equation (1) are also transformed: $a_0 = c(1+\delta a_1)$, $b_0 = (1+\delta b_1)/c$, $p_0 = 1+\delta p_1$, $q_0 = 1+\delta q_1$, $r = 1+\delta r_1$. The new independent variable is $\zeta = n\lambda_1$ where $\lambda = 1+\delta\lambda_1$. We thus find

$$\eta + \underline{\eta} = -\frac{\zeta + a_1}{z/c - 1} - \frac{\zeta + b_1}{cz - 1} \tag{2a}$$

$$z\bar{z} = \frac{(\eta - \tilde{\zeta} - p_1)(\eta - \tilde{\zeta} - q_1)}{(\eta - r_1)(\eta + r_1)} \tag{2b}$$

where the 'tilde' means a shift by half a lattice spacing. The constraint now becomes $a_1 + b_1 = p_1 + q_1$. Equation (2) is a discrete form of the Painlevé V equation. This can be verified by considering the continuous limit of (2). We put $\lambda_1 = \epsilon$ and at the limit $\epsilon \to 0$ the independent variable ζ goes over to t. We have $\eta = -t/(z-1) + \epsilon w$. The parameters $a_1 = \epsilon\alpha$, $b_1 = \epsilon\beta$, $p_1 = \epsilon\phi$, $q_1 = \epsilon\psi$, $r_1 = \epsilon\rho$ and $c = 1 + \epsilon\gamma$ satisfy the constraint $\alpha + \beta = \phi + \psi$. The resulting equation for z is precisely P_V:

$$z'' = \left(\frac{1}{2z} + \frac{1}{z-1}\right) z'^2 - \frac{z'}{t} + 2\rho^2 \frac{z(z-1)^2}{t^2} - \frac{(\phi-\psi)^2}{2}\frac{(z-1)^2}{zt^2} - 2\gamma(\alpha-\beta)\frac{z}{t} - 2\gamma^2\frac{z(z+1)}{z-1}. \tag{3}$$

From d-P_V we can obtain a form of d-P_{IV}. We put $z = \chi/\delta$ and similarly for the parameters $c = \delta$ and $a_1 = a_2/\delta^2$. At the limit $\delta \to 0$ we obtain

$$\eta + \underline{\eta} = -\frac{a_2}{\chi} - \frac{\zeta + b_1}{\chi - 1} \tag{4a}$$

$$\chi\bar{\chi} = -a_2\frac{\eta - \tilde{\zeta} - p_1}{\eta^2 - r_1^2} \tag{4b}$$

with the constraint $q_1 = a_1 + b_1 - p_1$. Equation (4) is indeed a d-P_{IV} as can be assessed by its continuous limit. Putting $\chi = -\alpha/\epsilon\eta + w$, $\zeta = \alpha(1/\epsilon + t)$, and $a_2 = \alpha/\epsilon$, $b_1 = \epsilon\beta$, $p_1 = 0$ and $r_1 = \epsilon\rho$ we find (at the limit $\epsilon \to 0$)

$$\eta'' = \frac{\eta'^2}{2\eta} + \frac{3}{2}\eta^3 - 2\alpha t\eta^2 + \left(\frac{\alpha^2 t^2}{2} + 2\alpha\beta + \alpha\right)\eta - \frac{2\alpha^2\rho^2}{\eta} \tag{5}$$

which is precisely P_{IV}. The two forms of d-P_V and d-P_{IV} are obtained here for the first time.

As is well known, d-P_V has another coalescence leading to d-P_{III}. From (2) we can perform this limit by putting $z = 1 + \delta\xi$ while the independent variable becomes $\zeta = \delta s$. The parameters of the equation are also transformed: $a_1 = \delta a_3$, $b_1 = -\delta a_3$, $p_1 = 1 + \delta p_2$, $q_1 = -1 - \delta p_2$, $c = 1 + \delta$, $r_1 = 1$. In the limit $\delta \to 0$ we obtain

$$\eta + \underline{\eta} = \frac{2s\xi + 2a_3}{1 - \xi^2} \tag{6a}$$

$$\xi + \bar{\xi} = \frac{2\tilde{s}\eta + 2p_2}{1 - \eta^2}. \tag{6b}$$

Equations (6) are indeed a discrete form of the Painlevé III equation. We have first obtained this equation in [11], where we have also given its continuous limit and its Lax pair. The study of its Schlesinger transformations in the framework of the 'Grand Scheme' [6] was presented in [5].

Both d-P_{IV} and d-P_{III} have coalescences that lead to d-P_{II}. We start with d-P_{IV} in the form (4) and put $\eta = 1 + \delta\theta$ and $\chi = 1 + \delta\omega/2$. The independent variable now becomes $\zeta = \delta^2\tau/2$. Moreover, we transform the parameters through $a_2 = -2 - \delta a_4$, $b_1 = \delta^2 b_2/2$, $p_1 = 1 + \delta^2 p_3/2$ and $r_1 = 1$. We thus find (at the limit $\delta \to 0$)

$$\theta + \underline{\theta} = a_4 - \omega - \frac{\tau + b_2}{\omega} \tag{7a}$$

$$\omega + \bar{\omega} = a_4 - \theta - \frac{\tilde{\tau} + p_3}{\theta}. \tag{7b}$$

Similarly, from d-P_{III} we introduce $\eta = 1 + \delta\theta$, $\xi = 1 + \delta\omega$, $s = -2 - \delta a_4 + \delta^2\tau$, $a_3 = 2 + \delta a_4 + \delta^2 b_2$ and $p_2 = 2 + \delta a_4 + \delta^2 p_3$, and we obtain the same d-P_{II}, equation (7). The latter is an equation already obtained in [12] as the asymmetric form of d-P_I. Its Lax pair was first derived in [13]. We can easily show that this equation is in fact a discrete form of P_{II}. Putting $\omega = 1 + \epsilon w + \epsilon^2\rho$, $\theta = 1 - \epsilon w + \epsilon^2\rho$ (with $\rho = (w^2 - w' - t)/4$), and $a_4 = 2$, $b_2 = -\epsilon^3\alpha$, $p_3 = \epsilon^3\alpha$ and $\tau = -1 - \epsilon^2 t$ we find (at the limit $\epsilon \to 0$)

$$w'' = 2w^3 + 2wt + 4\alpha - 1. \tag{8}$$

At this point it would seem that a way to proceed in order to obtain a d-P_I from (7) would be through a symmetrization. However, this is not the proper coalescence procedure. A different approach is based on the relation of the discrete $\mathbb{P}$'s to the Schlesinger transformations of the continuous $\mathbb{P}$'s. Thus one would expect d-P_{IV} with have two different degenerations associated with two distinct coalescence patterns, since d-P_{IV} is related to P_V which has degenerations to both P_{III} and P_{IV} [14]. One of these degenerations was given above, leading to the d-P_{II} (7) itself related to P_{IV}. The second one can also be found.

Indeed, putting $\eta = r_1 + \delta y$, $\chi = \delta x$, $\zeta = \delta(z + \mu)$, $p_1 = r_1 - \delta\mu$, $a_2 = -2\delta^2 r_1$ and $b_1 = 2r_1$, we find in the limit $\delta \to 0$

$$y + \underline{y} = 2r_1 \left(\frac{1}{x} + x\right) + z + \mu \tag{9a}$$

$$x\bar{x} = 1 - \frac{\tilde{z}}{y}. \tag{9b}$$

These equations, rewritten in terms of x only, were first obtained in [7] and studied in detail in [15]. They are known under the name of the alternate-d-$\mathrm{P_{II}}$ equation.

Following the same spirit (the relation between d-$\mathbb{P}$'s and continuous $\mathbb{P}$'s), we expect both d-$\mathrm{P_{II}}$'s, namely (7) and alternate-d-$\mathrm{P_{II}}$ (9), to have a common degeneration through coalescence to a d-$\mathrm{P_I}$ related to continuous $\mathrm{P_{II}}$. In the case of alternate-d-$\mathrm{P_{II}}$ this degeneration is known under the name of alternate-d-$\mathrm{P_I}$ [7, 15]. Indeed, starting from (7) we take $a_4 = 2$, $\omega = 1 + \delta X$, $\tau = \delta^3 Z$, $\theta = \delta^2 Y$, $p_3 = 0$ and $b_2 = 1 - \delta^2 K$, and obtain at the $\delta \to 0$ limit

$$X + \overline{X} = -\frac{\tilde{Z}}{Y} \tag{10a}$$

$$Y + \underline{Y} = K - X^2. \tag{10b}$$

Similarly, we obtain the same equations from (9) through $x = 1 + \delta X$, $y = Y$, $z = \delta Z$, $r_1 = 1/2\delta^2$ and $\mu = K - 1/\delta^2$.

Let us summarize the degenerations through the coalescence cascade of q-$\mathrm{P_{VI}}$:

$$\begin{array}{ccccccc}
q\text{-}\mathrm{P_{VI}} & \longrightarrow & \text{d-}\mathrm{P_{V}} & \longrightarrow & \text{d-}\mathrm{P_{IV}} & \longrightarrow & \text{alt-d-}\mathrm{P_{II}} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \text{d-}\mathrm{P_{III}} & \longrightarrow & \text{d-}\mathrm{P_{II}} & \longrightarrow & \text{alt-d-}\mathrm{P_{I}}
\end{array}$$

The pattern is richer than that for continuous $\mathrm{P_{VI}}$ because the discrete (difference) equations partake of both the continuous and the discrete world.

All the equations that we have derived above are integrable. This can be assessed through the coalescence procedure since they are all degenerate forms of q-$\mathrm{P_{VI}}$. They all satisfy the singularity confinement criterion [16]. Finally, they do possess Lax pairs which (for simple readability reasons) are presented in the appendix.

In this paper we have presented the degeneration of q-$\mathrm{P_{VI}}$ through a process of coalescence of singularities and obtained the Lax pairs of the resulting equations. The forms of the Lax pairs, and in particular the data of singularities, help us understand the coincidence of the discrete Painlevé equations with the Schlesinger transformations of the continuous $\mathbb{P}$'s (a point that establishes the relation of this work to that of Jimbo and Miwa [8]).

The forms of d-$\mathrm{P_V}$ and d-$\mathrm{P_{IV}}$ are quite new. In the case of d-$\mathrm{P_V}$ it is interesting to point out that the same equation has just been obtained [17] in a totally different setting (starting from the similarity reduction of the lattice mKdV equation). This is the first time that a Lax pair has been obtained for a form of d-$\mathrm{P_V}$ and d-$\mathrm{P_{IV}}$. For instance, none is known for what are considered the 'standard' forms of q-$\mathrm{P_V}$ and d-$\mathrm{P_{IV}}$. (But, to be fair, we must stress the fact that these mappings are symmetrized forms of richer equations that might correspond to higher discrete Garnier systems.)

The fact that q-$\mathrm{P_{VI}}$ is a q-discrete equation and its relation to q-$\mathrm{P_{III}}$ indicate other possible paths of investigation. In [10] we have studied in detail the discrete forms obtained from q-$\mathrm{P_{III}}$ by methods other than degeneration through coalescence. Their richness is most promising since the same procedure applied to q-$\mathrm{P_{VI}}$ would lead to novel equations, in particular of q form.

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Appendix

In this section we shall present the results for the Lax pairs of the equations resulting from the degeneration of q-$\mathrm{P_{VI}}$. For completeness reasons, we start by recalling the Lax pair of q-$\mathrm{P_{VI}}$ (which was already been published in [1]).

A.1. Lax pair for q-P_{VI}

The basic equations are

$$Y(qx,t) = \frac{A(x,t)}{\kappa_1(x-ta_1)(x-a_3)} Y(x,t) \tag{A1}$$

with

$$A(x,t) = \begin{pmatrix} \kappa_1((x-y)(x-\alpha)+z_1) & \kappa_2 w(x-y) \\ \kappa_1 w^{-1}(\gamma x+\delta) & \kappa_2((x-y)(x-\beta)+z_2) \end{pmatrix}$$

$$Y(x,qt) = \frac{B(x,t)}{x-qta_2} Y(x,t) \tag{A2}$$

$$B(x,t) = xI + B_0(t).$$

The matrix $B_0(t) = (B_{ij})$ is parametrized as follows:

$$B_{11} = \frac{-q\kappa_2\bar{z}}{1-\kappa_2\bar{z}}\left(-\beta + \frac{t(a_1+a_2)-y}{\kappa_2\bar{z}}\right)$$

$$B_{22} = \frac{-q\kappa_1\bar{z}}{1-q\kappa_1\bar{z}}\left(-\bar{\alpha} + \frac{qt(a_1+a_2)-\bar{y}}{q\kappa_1\bar{z}}\right)$$

$$B_{12} = \frac{q\kappa_2\bar{z}}{1-\kappa_2\bar{z}} w$$

$$B_{21} = \frac{q\kappa_1\bar{z}}{w(1-q\kappa_1\bar{z})}\left(qta_1 - \bar{\alpha} + \frac{qta_2-\bar{y}}{q\kappa_1\bar{z}}\right)\left(ta_1 - \beta + \frac{ta_2-y}{\kappa_2\bar{z}}\right)$$

$$= \frac{q\kappa_1\bar{z}}{w(1-q\kappa_1\bar{z})}\left(qta_2 - \bar{\alpha} + \frac{qta_1-\bar{y}}{q\kappa_1\bar{z}}\right)\left(ta_2 - \beta + \frac{ta_1-y}{\kappa_2\bar{z}}\right).$$

Here

$$\alpha = \frac{1}{\kappa_1-\kappa_2}[y^{-1}((\theta_1+\theta_2)t - \kappa_1 z_1 - \kappa_2 z_2) - \kappa_2((a_1+a_2)t + a_3 + a_4 - 2y)]$$

$$\beta = \frac{1}{\kappa_1-\kappa_2}[-y^{-1}((\theta_1+\theta_2)t - \kappa_1 z_1 - \kappa_2 z_2) + \kappa_1((a_1+a_2)t + a_3 + a_4 - 2y)]$$

$$\gamma = z_1 + z_2 + (y+\alpha)(y+\beta) + (\alpha+\beta)y - a_1a_2t^2 - (a_1+a_2)(a_3+a_4)t - a_3a_4$$

$$\delta = y^{-1}(a_1a_2a_3a_4t^2 - (\alpha y + z_1)(\beta y + z_2))$$

and

$$z_1 = \frac{(y - ta_1)(y - ta_2)}{q\kappa_1 z} \qquad z_2 = q\kappa_1(y - a_3)(y - a_4)z.$$

The compatibility condition

$$A(x, qt)B(x, t) = B(qx, t)A(x, t)$$

leads to q-P$_{\rm VI}$

$$\frac{y\bar{y}}{a_3a_4} = \frac{(\bar{z} - tb_1)(\bar{z} - tb_2)}{(\bar{z} - b_3)(\bar{z} - b_4)}$$
$$\frac{z\bar{z}}{b_3b_4} = \frac{(y - ta_1)(y - ta_2)}{(y - a_3)(y - a_4)}$$
$$\frac{\bar{w}}{w} = \frac{b_4}{b_3}\frac{\bar{z} - b_3}{\bar{z} - b_4}$$

where

$$b_1 = \frac{a_1a_2}{\theta_1} \qquad b_2 = \frac{a_1a_2}{\theta_2} \qquad b_3 = \frac{1}{q\kappa_1} \qquad b_4 = \frac{1}{\kappa_2}.$$

A.2. Degeneration q-P$_{VI}$ → d-P$_V$

We adopt the normalization

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ \theta_1 & \theta_2 & \kappa_1 & \kappa_2 \end{pmatrix} = \begin{pmatrix} a & 1/a & c & 1/c \\ 1/\theta & \theta & 1/q\kappa & \kappa \\ \theta & 1/\theta & \kappa & 1/\kappa \end{pmatrix}.$$

The equation d-P$_{\rm V}$ is obtained through $z = p^{-z_0-\lambda}$, $w = 1 - p^{w_0}$, $a = p^{a_0}/c$, $\kappa = p^{\kappa_0}$, $\theta = p^{\theta_0}$, $q = p^{\lambda}$, $t = p^{\zeta}$, $(\zeta = n\,\lambda)$ and $\delta = 1 - p \to 0$ (and elimination of w_0 through gauge transformation),

$$\frac{\mathrm{d}}{\mathrm{d}x}Y(x, n) = A(x, n)\,Y(x, n) \tag{A3}$$
$$Y(x, n + 1) = B(x, n)Y(x, n) \tag{A4}$$

where

$$A(x, n) = \frac{A^0}{\lambda x} + \frac{A^+}{\lambda(x - c)} + \frac{A^-}{\lambda(x - 1/c)}$$

with

$$A^0_{11} = -\frac{1}{2\kappa_0}\left((z_0 - \kappa_0)(y - 1/c) - \frac{\zeta + a_0}{c}\right)((z_0 - \kappa_0)(y - c) - c(\zeta - a_0))$$
$$+\frac{1}{2}(z_0 + \zeta + \kappa_0 - 2a_0) + \frac{(\theta_0 + a_0)(\theta_0 - a_0)}{2\kappa_0}$$
$$A^0_{22} = \frac{1}{2\kappa_0}\left((z_0 - \kappa_0)(y - 1/c) - \frac{\zeta + a_0}{c}\right)((z_0 - \kappa_0)(y - c) - c(\zeta - a_0))$$
$$-\frac{1}{2}(z_0 + \zeta + \kappa_0 + 2a_0) - \frac{(\theta_0 + a_0)(\theta_0 - a_0)}{2\kappa_0}$$
$$A^0_{12} = y$$
$$A^0_{21} = -\frac{1}{4\kappa_0^2 y}\Bigg[\left((z_0 - \kappa_0)(y - 1/c) - \frac{\zeta + a_0}{c}\right)((z_0 - \kappa_0)(y - c) - c(\zeta - a_0))$$

$$-\kappa_0(z_0+\zeta)+(a_0-\kappa_0+\theta_0)(a_0+\kappa_0-\theta_0)\Big]$$
$$\times\Bigg[\left((z_0-\kappa_0)(y-1/c)-\frac{\zeta+a_0}{c}\right)((z_0-\kappa_0)(y-c)-c(\zeta-a_0))$$
$$-\kappa_0(z_0+\zeta)+(a_0-\kappa_0-\theta_0)(a_0+\kappa_0+\theta_0)\Bigg]$$

$$A^+_{11}=a_0-\zeta-\frac{y-c}{2\kappa_0(c-1)(c+1)y}\Bigg[(a_0-\kappa_0+\theta_0)(a_0-\kappa_0-\theta_0)$$
$$+\left((z_0-\kappa_0)(y-1/c)-\frac{\zeta+a_0}{c}\right)((z_0-\kappa_0)(y-c)-c(\zeta-2\kappa_0-a_0))\Bigg]$$

$$A^+_{22}=\frac{y-c}{2\kappa_0(c-1)(c+1)y}\Bigg[(a_0-\kappa_0+\theta_0)(a_0-\kappa_0-\theta_0)$$
$$+\left((z_0-\kappa_0)(y-1/c)-\frac{\zeta+a_0}{c}\right)((z_0-\kappa_0)(y-c)-c(\zeta+2\kappa_0-a_0))\Bigg]$$

$$A^+_{12}=\frac{y-c}{(c-1)(c+1)}$$

$$A^+_{21}=\frac{(c-1)(c+1)}{y-c}A^+_{11}A^+_{22}$$

$$A^-_{11}=\frac{c^2(y-1/c)}{2\kappa_0(c-1)(c+1)y}\Bigg[(a_0+\kappa_0+\theta_0)(a_0+\kappa_0-\theta_0)$$
$$+\left((z_0-\kappa_0)(y-1/c)-\frac{\zeta+2\kappa_0+a_0}{c}\right)((z_0-\kappa_0)(y-c)-c(\zeta-a_0))\Bigg]$$

$$A^-_{22}=a_0+\zeta-\frac{c^2(y-1/c)}{2\kappa_0(c-1)(c+1)y}\Bigg[(a_0+\kappa_0+\theta_0)(a_0+\kappa_0-\theta_0)$$
$$+\left((z_0-\kappa_0)(y-1/c)-\frac{\zeta+2\kappa_0+a_0}{c}\right)((z_0-\kappa_0)(y-c)-c(\zeta-a_0))\Bigg]$$

$$A^-_{12}=\frac{c^2(y-1/c)}{(c-1)(c+1)}$$

$$A^-_{21}=\frac{(c-1)(c+1)}{c^2(y-1/c)}A^-_{11}A^-_{22}$$

and

$$A^\infty=-A^0-A^+-A^-=\begin{pmatrix}-\kappa_0 & 0\\ 0 & \kappa_0\end{pmatrix}.$$

The eigenvalues of A^0/λ are $(-a_0+\theta_0)/\lambda$, $(-a_0-\theta_0)/\lambda$, and those of $A^\pm/\lambda$ are 0, $(a_0/\lambda)\mp n$. The matrix B is given by

$$B_{11}=\frac{1}{x-c}[(z_0+\zeta+\theta_0)(z_0+\zeta-\theta_0)y^{-1}-(z_0+\kappa_0)(z_0+\zeta-\kappa_0+a_0)c^{-1}$$
$$-(z_0+\kappa_0)(z_0-\zeta+\kappa_0-a_0)c+(z_0-\kappa_0)(z_0+\kappa_0)y$$
$$+2\kappa_0(\bar z_0+\kappa_0+\lambda)(x-c^{-1}-c+y)]$$

$$B_{22}=\frac{1}{\bar y(x-c)}\Bigg[2\kappa_0(\bar z_0-\kappa_0)x-(a_0+\theta_0)(a_0-\theta_0)-\kappa_0(\bar z_0+\bar\zeta-\kappa_0)$$
$$-\left((\bar z_0-\kappa_0)(\bar y-1/c)-\frac{\bar\zeta+a_0}{c}\right)((\bar z_0-\kappa_0)(\bar y-c)-c(\bar\zeta-a_0))\Bigg]$$

$$B_{12} = -\frac{2\kappa_0}{x-c}$$

$$B_{21} = \frac{1}{2\kappa_0 y\bar{y}(x-c)}\Bigg[(a_0+\kappa_0+\theta_0)(a_0+\kappa_0-\theta_0) + \left((\bar{z}_0-\kappa_0)(\bar{y}-1/c) - \frac{\bar{\zeta}+2\kappa_0+a_0}{c}\right)((\bar{z}_0-\kappa_0)(\bar{y}-c)-c(\bar{\zeta}-a_0))\Bigg] \times\Bigg[(a_0-\kappa_0+\theta_0)(a_0-\kappa_0-\theta_0)+2(\bar{z}_0-z_0+\lambda)(y-c) + \left((z_0+\kappa_0)(y-1/c)-\frac{\zeta-2\kappa_0+a_0}{c}\right)((z_0+\kappa_0)(y-c)-c(\zeta-a_0))\Bigg].$$

The compatibility condition

$$\frac{\mathrm{d}}{\mathrm{d}x}B(x,n) = A(x,n+1)B(x,n) - B(x,n)A(x,n) \tag{A5}$$

leads to d-P_{V} in the form

$$\bar{z}_0 + z_0 + \lambda = \frac{\zeta+a_0}{yc-1} + \frac{\zeta-a_0}{y/c-1}$$

$$y\bar{y} = \frac{(\bar{z}_0+\bar{\zeta}+\theta_0)(\bar{z}_0+\bar{\zeta}-\theta_0)}{(\bar{z}_0-\kappa_0)(\bar{z}_0+\lambda+\kappa_0)}.$$

A.3. Degeneration d-P_V → d-P_{IV}

To get d-P_{IV}, we need a change of variable $x \to x/c$ and a gauge transformation. This is obtained through $n \to n + \beta/\lambda + \delta^{-2}\gamma_1/2\lambda$, $y = y_1/\delta$, $c = \delta$, $\theta_0 = \delta^{-2}\gamma_1/2$, $a_0 = \alpha_1 - \beta_1 - \delta^{-2}\gamma_1/2$ and $\delta \to 0$.

The system of deformation equations is (A3) and (A4), where

$$A(x,n) = \frac{A^{0,2}}{\lambda x^2} + \frac{A^0}{\lambda x} + \frac{A^1}{\lambda(x-1)}$$

with

$$A_{11}^{0,2} = \frac{1}{2\kappa_0}((z_0-\kappa_0)(y_1-1)-\zeta-\alpha_1)((z_0-\kappa_0)y_1-\gamma_1) - \frac{\gamma_1(\alpha_1-\beta_1+\kappa_0)}{2\kappa_0}$$

$$A_{22}^{0,2} = -\frac{1}{2\kappa_0}((z_0-\kappa_0)(y_1-1)-\zeta-\alpha_1)((z_0-\kappa_0)y_1-\gamma_1) + \frac{\gamma_1(\alpha_1-\beta_1-\kappa_0)}{2\kappa_0}$$

$$A_{12}^{0,2} = -y_1$$

$$A_{21}^{0,2} = -\frac{1}{y_1}A_{11}^{0,2}A_{22}^{0,2}$$

$$A_{11}^{1} = -\frac{y_1-1}{2\kappa_0 y_1}[((z_0-\kappa_0)y_1-z_0-\zeta-\alpha_1-\kappa_0)((z_0-\kappa_0)y_1-\gamma_1)-\gamma_1(\alpha_1-\beta_1+\kappa_0)]$$

$$A_{22}^{1} = \frac{y_1-1}{2\kappa_0 y_1}[((z_0-\kappa_0)y_1-z_0-\zeta-\alpha_1-\kappa_0)((z_0-\kappa_0)y_1-\gamma_1)-\gamma_1(\alpha_1-\beta_1+\kappa_0)] +\alpha_1+\zeta$$

$$A_{12}^{1} = y_1 - 1$$

$$A_{21}^{1} = \frac{1}{y_1-1}A_{11}^{1}A_{22}^{1}$$

$$A^0 = -A^1 + \begin{pmatrix} \kappa_0 & 0 \\ 0 & -\kappa_0 \end{pmatrix} \qquad \text{and} \qquad A^\infty = -A^0 - A^1 = \begin{pmatrix} -\kappa_0 & 0 \\ 0 & \kappa_0 \end{pmatrix}.$$

The eigenvalues of A^1/λ are 0, $(\alpha_1/\lambda)+n$, and those of $A^{0,2}/\lambda$ are 0, $-(\gamma_1/\lambda)$. When $A^{0,2}$ is diagonalized by a gauge transformation with C as

$$C^{-1}A^{0,2}C/\lambda = \begin{pmatrix} -\gamma_1/\lambda & 0 \\ 0 & 0 \end{pmatrix}$$

the diagonal parts of $C^{-1}A^0C/\lambda$ are

$$\begin{pmatrix} -\beta_1/\lambda - n & * \\ * & (\beta_1 - \alpha_1)/\lambda \end{pmatrix}.$$

The matrix B is given by

$$B_{11} = \frac{1}{x}[\gamma_1(z_0+\zeta+\beta_1){y_1}^{-1} - (z_0+\kappa_0)(z_0+\zeta+\alpha_1+\gamma_1-\kappa_0) + (z_0-\kappa_0)(z_0+\kappa_0)y_1$$
$$+2\kappa_0(\bar{z}_0+\kappa_0+\lambda)(x+y_1-1)]$$
$$B_{22} = 2\kappa_0(\bar{z}_0-\kappa_0) - \frac{((\bar{z}_0-\kappa_0)(\bar{y}_1-1) - \bar{\zeta} - \alpha_1)((\bar{z}_0-\kappa_0)\bar{y}_1 - \gamma_1) - \gamma_1(\alpha_1-\beta_1-\kappa_0)}{x}$$
$$B_{12} = -2\kappa_0$$
$$B_{21} = \frac{1}{2\kappa_0^2 y_1\bar{y}_1}[\gamma_1(z_0+\zeta+\beta_1){y_1}^{-1}$$
$$-(z_0+\kappa_0)(z_0+\zeta+\alpha_1+\gamma_1-\kappa_0) + (2\kappa_0(\bar{z}_0+\lambda) + {z_0}^2 + {\kappa_0}^2)y_1]$$
$$\times[((\bar{z}_0-\kappa_0)\bar{y}_1 - \bar{z}_0 - \bar{\zeta} - \alpha_1 - \kappa_0)((\bar{z}_0-\kappa_0)\bar{y}_1 - \gamma_1) - \gamma_1(\alpha_1-\beta_1+\kappa_0)].$$

The compatibility condition is (A5), and we obtain the following expression for d-P_{IV}

$$\bar{z}_0 + z_0 + \lambda = \frac{\zeta+\alpha_1}{y_1-1} + \frac{\gamma_1}{y_1}$$
$$y_1\bar{y}_1 = \frac{\gamma_1(\bar{z}_0+\bar{\zeta}+\beta_0)}{(\bar{z}_0-\kappa_0)(\bar{z}_0+\lambda+\kappa_0)}.$$

A.4. Degeneration d-P_V → d-P_{III}

To obtain d-P_{III}, we need a change of variable $x \to 1+\delta x$ and a gauge transformation. This is obtained through $y = 1+\delta y_0$, $\lambda = \delta\lambda_0$ $(s = n\lambda_0)$, $c = 1+\delta$, $\kappa_0 = 1$, $a_0 = \delta\alpha_0$, $\theta_0 = 1-\delta\beta_0$ and $\delta \to 0$. The system of deformation equations is (A3) and (A4), where

$$A(x,n) = \frac{A^{\infty,2}}{\lambda_0} + \frac{A^+}{\lambda_0(x-1)} + \frac{A^-}{\lambda_0(x+1)}$$
$$A^{\infty,2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$A^+ = \frac{(y_0-1)(y_0+1)}{2}\begin{pmatrix} z_0 - 1 + \frac{s+\beta_0}{y_0+1} + \frac{2\beta_0+2\alpha_0}{(y_0-1)(y_0+1)} & -\frac{1}{y_0-1} \\ A_{21}^+ & -z_0+1-\frac{s+\beta_0}{y_0-1} \end{pmatrix}$$

with

$$A_{21}^+ = (y_0-1)\left(z_0 - 1 + \frac{s+\beta_0}{y_0+1} + \frac{2\beta_0+2\alpha_0}{(y_0-1)(y_0+1)}\right)\left(z_0 - 1 + \frac{s+\beta_0}{y_0-1}\right)$$

$$A^- = \frac{(y_0-1)(y_0+1)}{2}\begin{pmatrix} -z_0+1-\dfrac{s+\beta_0}{y_0+1} & \dfrac{1}{y_0+1} \\ A_{21}^- & z_0-1+\dfrac{s+\beta_0}{y_0-1}+\dfrac{-2\beta_0+2\alpha_0}{(y_0-1)(y_0+1)} \end{pmatrix}$$

$$A_{21}^- = -(y_0+1)\left(z_0-1+\frac{s+\beta_0}{y_0-1}+\frac{-2\beta_0+2\alpha_0}{(y_0-1)(y_0+1)}\right)\left(z_0-1+\frac{s+\beta_0}{y_0+1}\right)$$

$$A^\infty = -A^+ - A^- = \begin{pmatrix} -\beta_0-\alpha_0 & 1 \\ A_{21}^\infty & \beta_0-\alpha_0 \end{pmatrix}$$

$$A_{21}^\infty = ((z_0-1)y_0+s+\beta_0)((z_0-1)y_0+s-\beta_0)-(z_0-1)(z_0-2\alpha_0-1).$$

The eigenvalues of $A^\pm/\lambda_0$ are 0, $(\alpha_0/\lambda_0)\mp n$. The matrix B is given by

$$B(x,n) = \frac{1}{x-1}\begin{pmatrix} (x-y_0)(\bar{z}_0+1)-s+\beta_0 & -1 \\ B_{21} & (x-\bar{y}_0)(\bar{z}_0-1)-\bar{s}-\beta_0 \end{pmatrix}$$

with

$$B_{21} = -((\bar{z}_0-1)(\bar{y}_0+1)+\bar{s}+\beta_0)((\bar{z}_0+1)(y_0+1)+s-\beta_0).$$

The compatibility condition is (A5), and leads to the following expression for d-P_{III}:

$$\bar{z}_0+z_0 = 2\frac{sy_0+\alpha_0}{1-y_0^2}$$
$$\bar{y}_0+y_0 = 2\frac{\tilde{s}\bar{z}_0+\beta_0+\lambda_0/2}{1-\bar{z}_0^2}.$$

A.5. Degeneration d-P_{IV} → d-P_{II}

To obtain d-P_{II}, we need a change of variable $x\to 1/(1+\delta x/2)$, a change of the dependent variable $A\to I/\lambda+A$ and a gauge transformation. This is obtained through $y_1 = 1+\delta y_2/2$, $z_0 = -1-\delta z_2$, $\lambda = \delta^2\lambda_2/2$ ($\tau = n\lambda_2$), $\kappa_0 = 1$, $\alpha_1 = \delta^2\alpha_2/2$, $\beta_1 = 1+\delta^2\beta_2/2$, $\gamma_1 = -2-\delta\gamma_2$ and $\delta\to 0$. The system of deformation equations is (A3) and (A4), where

$$A(x,n) = \frac{A^{\infty,3}}{\lambda_2}x+\frac{A^{\infty,2}}{\lambda_2}+\frac{A^0}{\lambda_2 x}$$

$$A^{\infty,3} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A^{\infty,2} = \begin{pmatrix} 0 & 1 \\ (\alpha_2-\beta_2+y_2(y_2+z_2-\gamma_2)) & -\gamma_2 \end{pmatrix}$$

$$A^0 = \begin{pmatrix} -y_2(y_2+z_2-\gamma_2) & y_2 \\ -(y_2+z_2-\gamma_2)(y_2(y_2+z_2-\gamma_2)+\alpha_2+\tau) & y_2(y_2+z_2-\gamma_2)+\alpha_2+\tau \end{pmatrix}.$$

The eigenvalues of A^0/λ_2 are 0, $(\alpha_2/\lambda_2)+n$. The matrix B is given by

$$B(x,n) = \begin{pmatrix} -\bar{z}_2 & -1 \\ -\bar{z}_2(\bar{y}_2+\bar{z}_2-\gamma_2) & x-(\bar{y}_2+\bar{z}_2-\gamma_2) \end{pmatrix}.$$

The compatibility condition is (A5), and leads to the following expression for d-P_{II}:

$$\bar{z}_2+z_2 = \gamma_2-y_2-\frac{\tau+\alpha_2}{y_2}$$
$$\bar{y}_2+y_2 = \gamma_2-\bar{z}_2-\frac{\tau+\beta_2}{\bar{z}_2}.$$

A.6. Degeneration d-P$_{III}$ → d-P$_{II}$

To find d-P$_{\rm II}$, we need a change of variable $x \to -1+\delta x$, a change of the dependent variable $A \to I/\lambda_0 + A$ and a gauge transformation. This is obtained through $n \to n-\delta^{-1}2/\lambda_2-\delta^{-2}\gamma_2/\lambda_2$, $y_0 = 1+\delta y_2$, $z_0 = 1+\delta z_2$, $\lambda_0 = \delta^2\lambda_2$ ($\tau = n\lambda_2$), $\alpha_0 = 2+\delta\gamma_2+\delta^2\alpha_2$, $\beta_0 = 2+\delta\gamma_2+\delta^2\beta_2$ and $\delta \to 0$. The system of deformation equations is (A3) and (A4), where

$$A(x,n) = \frac{A^{\infty,3}}{\lambda_2}x + \frac{A^{\infty,2}}{\lambda_2} + \frac{A^0}{\lambda_2 x}$$

$$A^{\infty,3} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A^{\infty,2} = \begin{pmatrix} -\gamma_2 & 1 \\ -(y_2z_2+\beta_2+\tau) & 0 \end{pmatrix}$$

$$A^0 = \begin{pmatrix} -y_2z_2 & y_2 \\ -z_2(y_2z_2+\alpha_2+\tau) & y_2z_2+\alpha_2+\tau \end{pmatrix}.$$

The eigenvalues of A^0/λ_2 are 0, $(\alpha_2/\lambda_2)+n$. The matrix B is given by

$$B(x,n) = \begin{pmatrix} -x+(y_2+\bar{z_2}-\gamma_2) & 1 \\ \bar{z_2}(y_2+\bar{z_2}-\gamma_2) & \bar{z_2} \end{pmatrix}.$$

The compatibility condition is (A5), and leads to the same d-P$_{\rm II}$ equations.

A.7. Degeneration d-P$_{IV}$ → alt.d-P$_{II}$

To obtain alt.d-P$_{\rm II}$, we need a change of variable $x \to \delta x$, a change of the dependent variable $A \to -I/\lambda x + A$ and elimination of w_1 through a gauge transformation. This is obtained through $y_1 = \delta y_3$, $z_0 = 1+\delta(2/\gamma_3)z_3$, $\lambda = \delta(2/\gamma_3)\lambda_3$ ($\sigma = n\lambda_3$), $\kappa_0 = 1$, $\alpha_1 = -2+\delta(2/\gamma_3)\alpha_3$, $\beta_1 = -1+\delta(2/\gamma_3)\beta_3$, $\gamma_1 = 2\delta^2$ and $\delta \to 0$. The system of deformation equations is (A3) and (A4), where

$$A(x,n) = \frac{A^{0,2}}{\lambda_3 x^2} + \frac{A^0}{\lambda_3 x} + \frac{A^{\infty,2}}{\lambda_3}$$

$$A^{\infty,2} = \begin{pmatrix} 0 & 0 \\ 0 & \gamma_3 \end{pmatrix}$$

$$A^0 = \begin{pmatrix} 0 & 1 \\ z_3(z_3+\sigma+\alpha_3)-\gamma_3\dfrac{z_3+\sigma+\beta_3}{y_3} & \sigma+\alpha_3 \end{pmatrix}$$

$$A^{0,2} = \begin{pmatrix} y_3z_3-\gamma_3 & -y_3 \\ z_3(y_3z_3-\gamma_3) & -y_3z_3 \end{pmatrix}.$$

The eigenvalues of $A^{0,2}/\lambda_3$ are 0, γ_3/λ_3. When $A^{0,2}$ is diagonalized by a gauge transformation using C so that

$$C^{-1}A^{0,2}C/\lambda_3 = \begin{pmatrix} -\gamma_3/\lambda_3 & 0 \\ 0 & 0 \end{pmatrix}$$

$C^{-1}A^0C/\lambda_3$ becomes

$$\begin{pmatrix} -\beta_3/\lambda_3 - n & \dfrac{z_3+\alpha_3-\beta_3}{\lambda_3} - \dfrac{\gamma_3}{\lambda_3 y_3} \\ \dfrac{z_3+\beta_3}{\lambda_3}+n & \dfrac{\beta_3-\alpha_3}{\lambda_3} \end{pmatrix}.$$

The matrix B is given by

$$B(x,n) = \begin{pmatrix} \gamma_3 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \bar{z}_3 + \bar{\sigma} + \alpha_3 - \gamma_3 y_3 & 1 \\ \bar{z}_3(\bar{z}_3 + \bar{\sigma} + \alpha_3) - \gamma_3 \dfrac{\bar{z}_3 + \bar{\sigma} + \beta_3}{\bar{y}_3} & \bar{z}_3 \end{pmatrix} \frac{1}{x}.$$

The compatibility condition is (A5), and leads to the following expression for alt.d-$\mathrm{P_{II}}$:

$$\bar{z}_3 + z_3 + \lambda_3 = \gamma_3 \left(y_3 + \frac{1}{y_3} \right) - (\sigma + \alpha_3)$$
$$\bar{y}_3 y_3 = 1 + \frac{\bar{\sigma} + \beta_3}{\bar{z}_3}.$$

A.8. Degeneration d-$P_{II} \to alt.d\text{-}P_I$

To find alt.d-$\mathrm{P_I}$ from d-$\mathrm{P_{II}}$ (the expression of A5), we need a change of variable $x \to \delta^{-3}(1+2^{1/3}\delta^2 x)$, a change of the dependent variable $A \to A - (\alpha_2+\tau)I/2\lambda_2 x + \gamma_2 I/2\lambda_2$ and a gauge transformation. This is obtained through $y_2 = -\delta^{-3}(1+2^{1/3}\delta^2 y_4)$, $z_2 = -2^{-1/3}\delta z_4$, $\alpha_2 = \delta^{-6} + 2^{-1/3}\delta^{-2}\alpha_3$, $\beta_2 = 0$, $\gamma_2 = -2\delta^{-3}$ and $\delta \to 0$. The system of deformation equations is (A3) and (A4), where

$$A(x,n) = \frac{A^{\infty,4}}{\lambda_2} x^2 + \frac{A^{\infty,3}}{\lambda_2} x + \frac{A^{\infty,2}}{\lambda_2}$$
$$A^{\infty,4} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$A^{\infty,3} = \begin{pmatrix} 0 & 1 \\ 4y_4^2 + 2z_4 + 2\alpha_4 & 0 \end{pmatrix}$$
$$A^{\infty,2} = \begin{pmatrix} -(2y_4^2 + z_4 + \frac{1}{2}\alpha_4) & -y_4 \\ y_4(4y_4^2 + 2z_4 + 2\alpha_4) - 2\tau & 2y_4^2 + z_4 + \frac{1}{2}\alpha_4 \end{pmatrix}.$$

The matrix B is given by

$$B(x,n) = \begin{pmatrix} 0 & -1 \\ 2\bar{z}_4 & 2(x + \bar{y}_4) \end{pmatrix}.$$

The compatibility condition is (A5), and leads to the following expression for alt.d-$\mathrm{P_I}$:

$$\bar{z}_4 + z_4 = -\alpha_4 - 2y_4^2$$
$$\bar{y}_4 + y_4 = -\frac{\tau}{\bar{z}_4}.$$

A.9. Degeneration alt.d-$P_{II} \to alt.d\text{-}P_I$

To get alt.d-$\mathrm{P_I}$ from alt.d-$\mathrm{P_{II}}$, we need a change of variable $x \to 1 + 2\delta x$, a change of the dependent variable $A \to A - \gamma_3(x^2-1)I/2\lambda_3 x^2 + (\sigma+\alpha_3)/2\lambda_3 x$ and gauge transformation. This is obtained through $y_3 = 1+\delta^2 y_4$, $z_3 = 2\delta^2 z_4$, $\lambda_3 = 4\delta^3\lambda_4(\tau = n\lambda_4)$, $\alpha_3 = -2+2\delta^2\alpha_4$, $\beta_3 = 0$, $\gamma_3 = -1$ and $\delta \to 0$. The system of deformation equations is (A3) and (A4), where

$$A(x,n) = \frac{A^{\infty,4}}{\lambda_4} x^2 + \frac{A^{\infty,3}}{\lambda_4} x + \frac{A^{\infty,2}}{\lambda_4}$$
$$A^{\infty,4} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$A^{\infty,3} = \begin{pmatrix} 0 & 1 \\ -2z_4 & 0 \end{pmatrix}$$

$$A^{\infty,2} = \begin{pmatrix} z_4 + \alpha_4/2 & -y_4 \\ -2y_4z_4 + 2\tau & -z_4 - \alpha_4/2 \end{pmatrix}.$$

The matrix B is given by

$$B(x, n) = \begin{pmatrix} -2(x + \bar{y}_4) & -1 \\ 2\bar{z}_4 & 0 \end{pmatrix}.$$

The compatibility condition is (A5), and leads to the same alt.d-P_I equations.

A.10. Summary

The discrete Painlevé equations which are obtained from degeneration of q-P_{VI} are considered as discrete deformation equations (Schlesinger transformations) for a linear problem $dY/dx = A(x)Y$ of the following form:

$$\text{alt.d-P}_\text{I}: \qquad A(x) = Ax^2 + Bx + C$$

$$\text{alt.d-P}_\text{II}: \qquad A(x) = A + \frac{B}{x} + \frac{C}{x^2}$$

$$\text{d-P}_\text{II}: \qquad A(x) = Ax + B + \frac{C}{x}$$

$$\text{d-P}_\text{IV},\ \text{d-P}_\text{III}: \qquad A(x) = A + \frac{B}{x-b} + \frac{C}{x-c}$$

$$\text{d-P}_\text{V}: \qquad A(x) = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}.$$

The types of singularities and the shifts of the monodromy data are as follows:

alt.d-P_I	alt.d-P_{II}	d-P_{II}	d-P_{III}	d-P_{IV}	d-P_V
4	2 + 2	3 + 1	2 + 1 + 1	2 + 1 + 1	1 + 1 + 1 + 1
$\begin{pmatrix} +1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} +1 & -1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & +1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & +1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & +1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} +1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Remarks. (1) The meaning of this list is as follows: the number in the first line expresses the multiplicity of each singularity (Poincaré rank + 1), and the numbers in the matrices are the increments of the monodromy data corresponding to each singularity.

At each singularity (of Poincaré rank r), a linear equation $dY/dx = A(x)Y$ has a unique formal solution of the form

$$Y(x) \sim G\hat{Y}(x)\,e^{T(x)}$$

$$\hat{Y}(x) = 1 + Y_1 \cdot (x - x_0) + \cdots$$

$$T(x) = \sum_{k=1}^{r} T_{-k} \frac{(x - x_0)^{-k}}{-k} + T_0 \log(x - x_0)\text{: diagonal.}$$

A monodromy preserving deformation transforms the monodromy data $T_0 = (t_{0,i}\delta_{ij})_{i,j=1,2}$ by integer increments only under the constraint (Fuchs' relation):

$$\sum_{\text{all singularities}} \operatorname{trace} T_0 = 0.$$

(2) From this table, we find that d-P_{III} is a composition of two d-P_{IV}'s of different directions:

$$\text{d-P}_\text{III} = \widetilde{\text{d-P}_\text{IV}} \circ \text{d-P}_\text{IV}$$

$$\begin{pmatrix} 0 & +1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} +1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \circ \begin{pmatrix} -1 & +1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

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